Calculus

Theorems

Squeeze Theorem pg 101 Stewart 8th edition

Given a function $f(x)$ such that when x is arbitrarily close to a then $g(x) \leq f(x) \leq h(x)$:
Example of alternating/oscillating function: $1 \leq sin(x) \leq 1$
If we want to find the $lim_{x \to a} f(x)$
then we can show that the limits of its bounds $g(x)$ and $h(x)$ are equal

1. So if we show that $\lim_{x \to a} g(x) = L$ and $\lim_{x \to a} h(x) = L$
2. then since $g(x) \leq f(x) \leq h(x)$, when x is close to a then $L \leq \lim_{x \to a} f(x) \leq L$
3. So then we can conclude that $\lim_{x \to a} f(x) = L$ by the squeeze theorem

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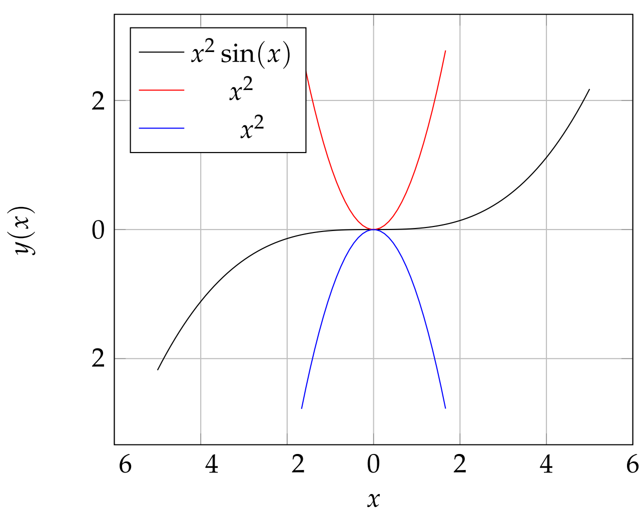
Example: Find $\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} x^2 \sin(x)$



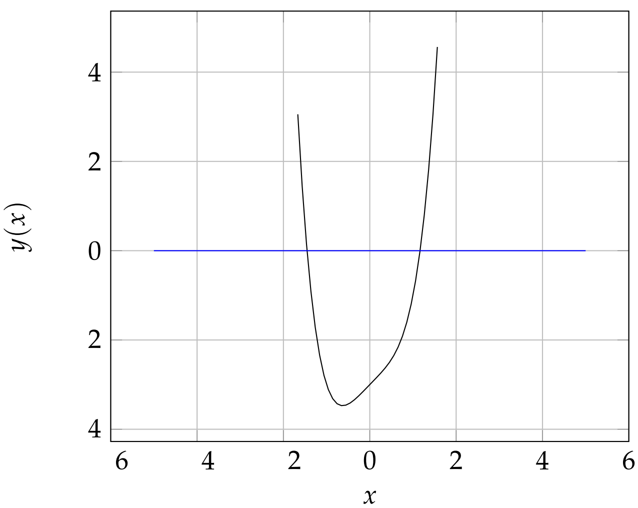
1. Establish your bounds: We know that $1 \leq sin(x) \leq 1$ so $x^2 \leq x^2 \sin(x) \leq x^2$
2. Take the limit of the endpoints: $\lim_{x \to 0}$ $x^2 = 0$ and $\lim_{x \to 0} x^2 = 0$
3. Conclude: So when x is arbitrarily close to 0 then $0 \leq \lim_{x \to 0} x^2 \sin(x) \leq 0$
   so then $\lim_{x \to 0} x^2 \sin(x) = 0$ by the squeeze theorem

Calculus

Intermediate Value Theorem pg 122 Stewart 8th edition

1. Show $f(x)$ is continuous on the closed interval $[a, b]$
2. let N be a number between $f(a)$ And $f(b)$ where $f(a) \neq f(b)$
3. Then there exists a number c in (a, b) such that $f(c) = N$
   This is commonly used to show that there exists a root of an equation in a certain interval, Let $N = 0$

Example: Show that there is a root to $f(x) = x^4 + x \quad 3$ in the interval $(1, 2)$



1. $f(x) = x^4 + x \quad 3$ is a polynomial therefore continuous on $[1, 2]$
2. Plug in endpoints: $f(1) = \quad 1$, which is negative $f(1) < 0$ and $f(2) = 15$, which is positive $f(2) > 0$
3. Since $f(1) < 0$ and $f(2) > 0$ there exists a point c in $(1, 2)$ such that $f(c) = 0$

Calculus

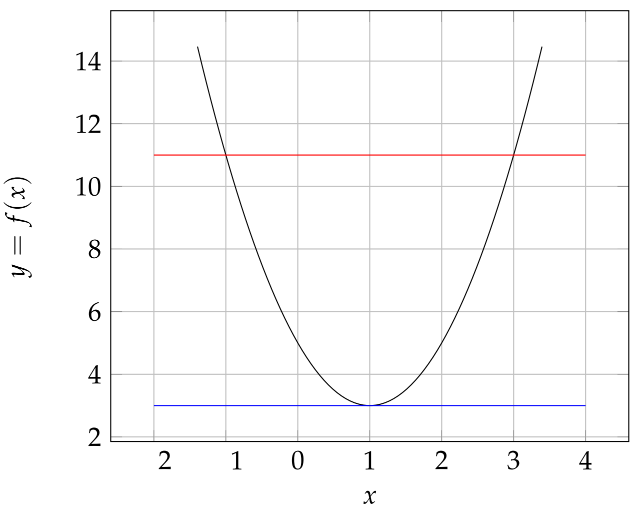
Rolle's Theorem pg 287 Stewart 8th edition

1. Show $f(x)$ is continuous on the closed interval $[a, b]$
   (that is find if there are any points of discontinuity in the given interval)

2. Show that $f(x)$is differentiable on the open interval (a, b)
   (continuous with no corners or cusps or any other immediate changes in slope)

3. Show $f(a) = f(b)$ then you can conclude there exists a number c in (a, b) such that $f'(c) = 0$

Example: Verify that there exists a critical point of $f(x) = 2x^2 \quad 4x + 5$ on the interval $[\quad 1, 3]$



1. $f(x)$ is a polynomial therefore continuous on $(\quad \infty, \infty)$

2. $f'(x) = 4x \quad 4$ exists for all x in $[\quad 1, 3]$, hence $f(x)$ differentiable

3. $f(\quad 1) = 11$ and $f(3) = 11$ so $f(\quad 1) = f(3)$
   so there exists a c in $(\quad 1, 3)$ such that $f'(c) = 0$ by Rolle's Theorem.
4. Find such a c such that $f'(c) = 0$, so $f'(x) = 4x \quad 4$.

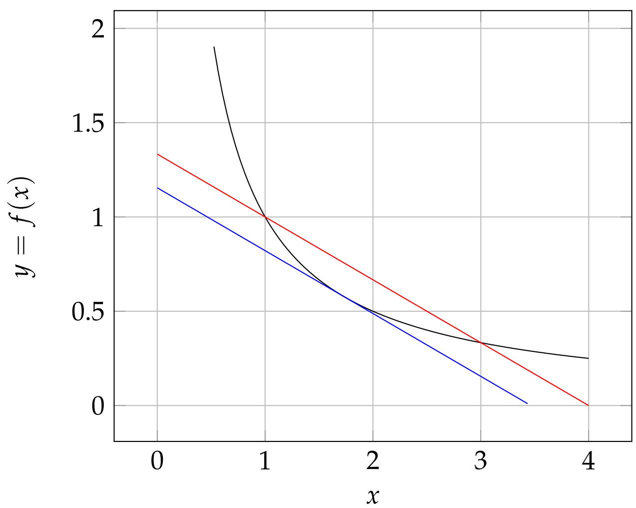
$$f'(x) = 0 \tag{1}$$
$$4x \quad 4 = 0 \tag{2}$$
$$4x = 4 \tag{3}$$
$$x = 1 \tag{4}$$

So $c = 1$, and $f'(c) = 4(1) \quad 4 = 4 \quad 4 = 0$✓ Observe that x=1 is in the interval $(\quad 1, 3)$

Mean Value Theorem pg 288 Stewart 8th edition

1. Show $f(x)$ is continuous on the closed interval $[a, b]$
2. Show $f(x)$ is differentiable on the open interval (a, b)
3. Then there exists a number c in (a, b) such that $f'(c) = \frac{f(b)\quad f(a)}{b\quad a}$ that is the tangent line to $(c, f(c))$ has the same slope as the secant through $(a, f(a))$ and $(b, f(b))$

Example: Show that there exists a point in the interval $[1, 3]$ such that the slope of the tangent line to that point is equal to the slope of the secant line through it's endpoints, on the curve of $f(x) = \frac{1}{x}$



1. The only discontinuity is at partition point $x = 0$ which is not in $[1, 3]$ so $f(x)$ is continuous on $[1, 3]$.

2. $f'(x) = \frac{1}{x^2}$ is only discontinuous at partition point $x = 0$ which is not in the interval $[1, 3]$ so $f(x)$ is differentiable on $(1, 3)$
   Plug in the endpoints and find the slope through them: $f(1) = 1$, $f(3) = \frac{1}{3}$ so $\frac{f(3)\quad f(1)}{3\quad 1} = \frac{1}{3}$

3. Then there exists a c in $[1, 3]$ such that $f'(c) = \frac{1}{3}$ so $f'(c) = \frac{1}{c^2} = \frac{1}{3}$ solving we get that $c = \pm\sqrt{3}$ since $+\sqrt{3}$ is in the interval $(1, 3)$ so c = $+\sqrt{3}$

L' Hospital's Rule pg 305 Stewart 8th edition

$$\text{L'Hospital's Rule}: \lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

If you are asked to take the limit of a rational equation of the form $\lim_{x\to a}\frac{f(x)}{g(x)}$

such that the limit is an ""Indeterminate Form" or of the form $\frac{\infty}{\infty}$ or $\frac{0}{0}$

1. First take the limit of the rational function and show that it is an indeterminate form:

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\infty}{\infty} \text{ or } \lim_{x\to a}\frac{f(x)}{g(x)} = \frac{0}{0}$$

2. Then take the derivative of the numerator and the denominator separately and take the limit:

$$\text{Find } \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

3. If the limit gos to a number N , 0 , or ∞ then that is your answer.

   if it gives another indeterminate form i.e. $\frac{\infty}{\infty}$ or $\frac{0}{0}$

   then repeat the process until the limit goes to a number, 0 or ∞.

Example: $\lim_{x\to\infty}\frac{ln(x)}{x}$

1. Show that the limit gives an indeterminate form: $\lim_{x\to\infty}\frac{ln(x)}{x} = \frac{\infty}{\infty}$
2. Take the derivative of the numerator and denominator separately. $\frac{dy}{dx}ln(x) = \frac{1}{x}$ and $\frac{dy}{dx}x = 1$
3. Take the limit of $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ $\quad \lim_{x\to\infty}\frac{1}{x} = \frac{1}{\infty} = 0$