Trigonometric Proofs

When proving a trigonometric identity we can use the following process as scratch work. Our goal is to have the left and right side look exactly the same.
Note: This process is not the only method that works. It may not be the shortest way either, but we should always reach a point where the sides are the same.

1. Change both sides of the equation to be in terms of sine and cosine.
2. If there are angle sums or differences, use the appropriate formulas to remove them.
3. If the angle is being multiplied by a constant, use appropriate formulas to remove it.
4. At this point we may need to combine, simplify, or expand terms. We should have an idea on what to do by examining both sides of the equation.
5. If there are any cosine terms with powers of 2 or greater, use the Pythagorean identities to change it to powers of sine.
6. Factor numerators and denominators. Cancel terms if possible.
7. At this point we should have that both sides are equal.

We have shown that the two sides are equal, but this is not a proper proof. To make this a proper and direct proof we must rewrite the steps in a new order. See the following examples for the process.

Example 1
Prove $\tan x \sin x = \sec x - \cos x$.

Step	Equation	Step
1 (Start)	$\tan x \sin x = \sec x - \cos x$	6 (End)
2	$\frac{\sin x}{\cos x} \sin x = \frac{1}{\cos x} - \cos x$	5
3	$\frac{\sin^2 x}{\cos x} = \frac{1-\cos^2 x}{\cos x}$	4
3	$\frac{\sin^2 x}{\cos x} = \frac{\sin^2 x}{\cos x}$	3

We now write the steps in order. We start with the left side and wrap around to the right side. Note that if a step repeats itself we do not write the equation again. The bottom row of the table should always repeat since we must get to a point where the left and right side are the same.

1. $\tan x \sin x$
2. $= \frac{\sin x}{\cos x} \sin x$
3. $= \frac{\sin^2 x}{\cos x}$
4. $= \frac{1-\cos^2 x}{\cos x}$
5. $= \frac{1}{\cos x} - \cos x$
6. $= \sec x - \cos x$

Example 2

Prove $\frac{\cot\theta+\tan\theta}{\tan(2\theta)} = \frac{1}{2}(\cot^2\theta - \tan^2\theta)$.

Step	Equation	Step
1 (Start)	$\frac{\cot\theta+\tan\theta}{\tan(2\theta)} = \frac{1}{2}(\cot^2\theta - \tan^2\theta)$	9 (End)
2	$\frac{\frac{\cos\theta}{\sin\theta}+\frac{\sin\theta}{\cos\theta}}{\frac{\sin(2\theta)}{\cos(2\theta)}} = \frac{1}{2}(\frac{\cos^2\theta}{\sin^2\theta} - \frac{\sin^2\theta}{\cos^2\theta})$	8
3	$\frac{\frac{\cos\theta}{\sin\theta}+\frac{\sin\theta}{\cos\theta}}{\frac{2\sin\theta\cos\theta}{\cos^2\theta-\sin^2\theta}} = \frac{1}{2}(\frac{\cos^2\theta}{\sin^2\theta} - \frac{\sin^2\theta}{\cos^2\theta})$	8
4	$(\frac{\cos^2\theta-\sin^2\theta}{2\sin\theta\cos\theta})(\frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta}) = \frac{1}{2}(\frac{\cos\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta})(\frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta})$	7
4	$(\frac{\cos^2\theta-\sin^2\theta}{2\sin\theta\cos\theta})(\frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta}) = \frac{1}{2}(\frac{\cos^2\theta}{\sin\theta\cos\theta} - \frac{\sin^2\theta}{\sin\theta\cos\theta})(\frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta})$	6
5	$(\frac{1-2\sin^2\theta}{2\sin\theta\cos\theta})(\frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta}) = (\frac{1-2\sin^2\theta}{2\sin\theta\cos\theta})(\frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta})$	5

1. $\frac{\cot\theta+\tan\theta}{\tan(2\theta)}$
2. $= \frac{\frac{\cos\theta}{\sin\theta}+\frac{\sin\theta}{\cos\theta}}{\frac{\sin(2\theta)}{\cos(2\theta)}}$
3. $= \frac{\frac{\cos\theta}{\sin\theta}+\frac{\sin\theta}{\cos\theta}}{\frac{2\sin\theta\cos\theta}{\cos^2\theta-\sin^2\theta}}$
4. $= (\frac{\cos^2\theta-\sin^2\theta}{2\sin\theta\cos\theta})(\frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta})$
5. $= (\frac{1-2\sin^2\theta}{2\sin\theta\cos\theta})(\frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta})$
6. $= \frac{1}{2}(\frac{\cos^2\theta}{\sin\theta\cos\theta} - \frac{\sin^2\theta}{\sin\theta\cos\theta})(\frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta})$
7. $= \frac{1}{2}(\frac{\cos\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta})(\frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta})$
8. $= \frac{1}{2}(\frac{\cos^2\theta}{\sin^2\theta} - \frac{\sin^2\theta}{\cos^2\theta})$
9. $= \frac{1}{2}(\cot^2\theta - \tan^2\theta)$